Indian Statistical Institute, Bangalore

B. Math (Hons.) Second Year

First Semester - Group Theory

Midterm Exam (supplementary) Total marks: 100 Date: 11th October 2024 Duration: 2 hours

Answer any Four. Each question carries 25 marks

1. (a) Prove that any group of prime order is cyclic (Marks: γ).

(b) Let H and K be subgroups of G. Prove that HK is a subgroup of G if and only if HK = KH

- 2. (a) Prove that there is a bijection between two right cosets of H.
 - (b) Prove that right cosets are distinct or identical (Marks: 7).

(c) Prove that H is a normal subgroup of G if and only if every right coset is a left coset (Marks: 10).

- 3. (a) State and prove third Sylow Theorem (Marks: 18).
 (b) Prove that a group of order pq is simple where p and q are prime.
- 4. (a) If H is a subgroup of S_n containing (1, 2) and $(1, 2, \dots, n)$, Prove $H = S_n$. (b) Find $\{\sigma(1, 2, 3)\sigma^{-1} \mid \sigma \in A_n\}$ for $n \ge 5$ (Marks: 10).
- 5. (a) Determine which subgroups of \mathbb{Z} can be written as a (internal-direct) product of its proper subgroups (Marks: 18).

(b) Suppose $G = A \ltimes_{\phi} B$. Prove that there is a normal subgroup H of G such that $H \simeq B$ and $G/H \simeq A$.